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THE NUMBER OF FUZZY SUBGROUPS OF CUBOID GROUPS $m \times 3 \times 2$

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Abstract

In this paper, we determine the number of fuzzy subgroups of cuboid groups $m \times 3 \times 2$ in size.

1. Introduction

Classification of fuzzy subgroups of finite groups is an important problem of fuzzy group theory. Many papers have dealt with finite groups. Groups of order one to order six have been studied by Filep [1], and Zhang and Zou [2] have determined the number of fuzzy subgroups of cyclic groups of order p^n, where p is a prime number. Murali and Makamba [3, 4] have determined the number of fuzzy subgroups of $Z_{p^n}, Z_p \times Z_p, Z_{p^n} \times Z_q$ and abelian groups of order $p^n q^m$, where p and q are different primes, and Tarnauceanu and Bentea [5] have determined this number for finite abelian groups.

The interesting result of Sulaiman [6] is a method to determine the

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number of fuzzy subgroups of finite groups. This method is called "*lattice method*". It can be used for abelian and non-abelian groups. By using this method, some formulas are determined. Sulaiman [6] has counted the number of fuzzy subgroups of *p*-*group* (Theorem 18), the number of fuzzy subgroups of $Z_p \times Z_q$ (Theorem 21) and the number of fuzzy subgroups of $Z_{p^n} \times Z_q$ (Theorem 23).

In [7], Sulaiman defined cuboid group and determined the number of fuzzy subgroups for some cuboid groups.

In this article, we construct the number of fuzzy subgroups of cuboid groups $m \times 2 \times 2$. Furthermore, we determine the number of fuzzy subgroups of cuboid groups $m \times 3 \times 2$. This formula is important to determine the number of fuzzy subgroups of cuboid groups in general, $m \times n \times s$ in size.

2. Cuboid Group

In this section, we summarize the preliminary definitions and theorems that are required later. We use some theorems of lattice method and prove these theorems which can be checked in [6].

Definition 1. Let X be a nonempty set. Then a fuzzy subset of X is a *function* from X into $[0, 1]$.

Definition 2 (Rosenfeld [8])**.** Let G be a group. Then a fuzzy subset μ of G is called a *fuzzy subgroup* of G if:

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in G$,

(2) $\mu(x^{-1}) \geq \mu(x), \forall x \in G$.

Theorem 1 (Sulaiman and Ghafur [9])**.** *A fuzzy subset* μ *of* G *is a fuzzy subgroup of* G *if and only if there is a chain of subgroups* G, $P_1(\mu) \leq P_2(\mu) \leq P_3(\mu) \leq \cdots \leq P_n(\mu) = G$ *such that* μ *is in the form*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu) \\ \theta_2, & x \in P_2(\mu) \\ \vdots \\ \theta_m, & x \in P_m(\mu). \end{cases} \tag{1}$$

If there is no confusion, then $P_i(\mu)$ as (1) can simply be written as P_i. We define *length* (or *order*) of fuzzy subgroup μ in (1) to be m.

Definition 3 (Sulaiman and Ghafur [9]). Let μ and γ be fuzzy subgroups of G of the forms

$$\mu(x) = \begin{cases} \theta_1, \ x \in A_1 \\ \theta_2, \ x \in A_2 \\ \vdots \\ \theta_m, \ x \in A_m \end{cases} \text{ and } \gamma(x) = \begin{cases} \delta_1, \ x \in B_1 \\ \delta_2, \ x \in B_2 \\ \vdots \\ \delta_n, \ x \in B_n \end{cases}$$

with $\theta_i, \delta_j \in [0, 1]$, $\theta_k > \theta_l$, $\delta_k > \delta_l$ for $k < l$ and $\bigcup_{1 \le i \le m} A_i = G$, $\bigcup_{1 \le j \le n} B_j = G$. Then we define that μ and γ are equivalent if $m = n$ and $A_i = B_i, \forall i \in \{1, 2, ..., n\}$.

It is easy to check that this relation is indeed an equivalence relation. Two fuzzy subgroups of G are said to be *different* if they are not equivalent.

We denote the number of fuzzy subgroups (different fuzzy subgroups) of G as $n(F_G)$, while the number of fuzzy subgroups μ of G with $P_1(\mu) = H$ is denoted by $n(F_{P_1 = H})$.

Theorem 2 (Sulaiman [6]). *Let $G_1 < G_2 < \cdots < G_{k-1} < G_k = G$ be the only maximal chain from G_1 to G on the lattice subgroups of G. Then*

$$n(F_{P_1 = G_1}) = \begin{cases} 1, & k = 1, 2, \\ 2^{k-2}, & k > 2. \end{cases}$$

Theorem 3 (Sulaiman and Prawoto [10, Theorem 3.3(ii)]). *Let G be a rectangle group with $m \in N$, $s = 2$. Then we have*:

(i) $n(F_{P_1 = K_2^m}) = 2.n(F_{P_1 = K_2^{m-1}}) + n(F_{P_1 = K_1^m})$.

(ii) $n(F_{P_1 = K_2^m}) = 2^m + (m-3)2^{m-2} = 2^{m-2}(m+1)$.

Definition 4. Let G be a group. Then a group G is called "*cuboid*" if the subgroups of G can be labeled by K_{ijk}, where $1 \leq i \leq m$, $1 \leq j \leq s$, $1 \leq k \leq t$ for some $m, s, t \in Z$ with $K_{111} = G$, $K_{m,s,t} = \{e\}$ and satisfying the following three conditions:

(i) for fixed i, j, $1 \leq i \leq m$, $1 \leq j \leq s$, $K_{ijk} < K_{ij(k-1)}$, $\forall k$, $2 \leq k \leq t$,

(ii) for fixed $i, k, 1 \leq i \leq m, 1 \leq k \leq s$, $K_{ijk} < K_{i(j-1)k}$, $\forall j, 2 \leq j \leq n$,

(iii) for fixed $j, k, 1 \leq j \leq n, 1 \leq k \leq s$, $K_{ijk} < K_{(i-1)jk}$, $\forall i, 2 \leq i \leq m$.

For this case, the size of the group is $m \times s \times t$. The diagram of lattice subgroups of cuboid group G is shown in Figure 1.

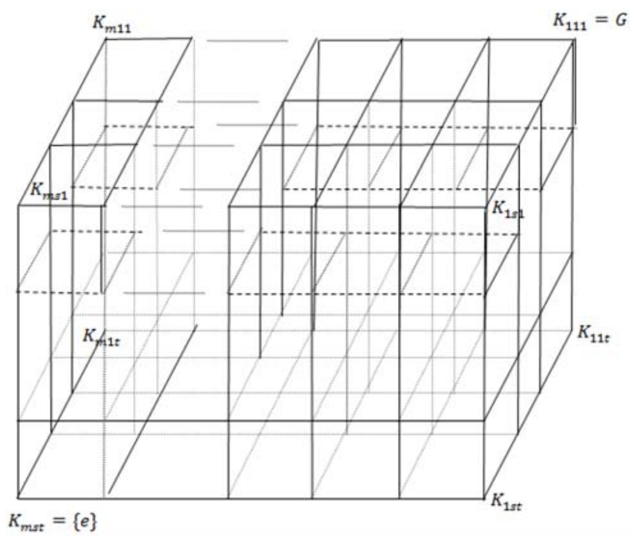

Figure 1. Lattice of cuboid group.

Theorem 4 (Sulaiman [7])**.** *Let G be a group satisfying Definition* 4 *with* $m \in N$, $n = s = 2$. *Then we have*:

(i) $n(F_{P_1=K_{m21}}) = 2.(F_{P_1=K_{(m-1)21}}) + 2^{m-2}$.

(ii) $n(F_{P_1=K_{m12}}) = 2.(F_{P_1=K_{(m-1)12}}) + 2^{m-2}$.

3. The Number of Fuzzy Subgroups of Cuboid Groups $m \times 3 \times 2$

First, we construct the formula for cuboid group $m \times 2 \times 2$.

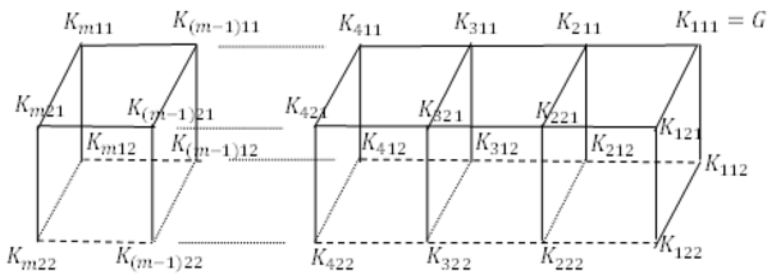

Figure 2. Lattice of cuboid group, $m \times 2 \times 2$.

Theorem 5. *Let G be a group satisfying Definition* 4 *with* $m \in N$, $n = s = 2$. *Then we have*:

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j22}}) + (2m+1)2^{m-1}.$$

Proof. By considering the diagram of lattice G (see Figure 2), we get

$$n(F_{P_1=K_{i21}}) = n(F_{P_1=K_{i12}}), \ \forall i \in \{1, 2, ..., m\} \tag{2}$$

and

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1} [n(F_{P_1=K_{j22}}) + n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j12}})] + n(F_{P_1=K_{m21}}) + n(F_{P_1=K_{m12}}) + n(F_{P_1=K_{m11}}). \tag{3}$$

Using lattice method (see [6]), we get

$$n(F_{P_1=K_{m21}}) = \sum_{j=1}^{m-1} [n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})] + n(F_{P_1=K_{m11}}). \tag{4}$$

According to (2), (3) can be written as

$$n(F_{P_1=K_{m12}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})]$$

$$+ 2.n(F_{P_1=K_{m21}}) + 2^{m-2}.$$

By using (4), we have

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + 2.n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})]$$

$$+ 2.\left(\sum_{j=1}^{m-1}[n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})] + n(F_{P_1=K_{m11}})\right)$$

$$+ 2^{m-2},$$

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + 4.n(F_{P_1=K_{j21}}) + 3.n(F_{P_1=K_{j11}})]$$

$$+ 3.2^{m-2},$$

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + 4.n(F_{P_1=K_{j21}})]$$

$$+ 3.[n(F_{P_1=K_{111}}) + n(F_{P_1=K_{211}}) + n(F_{P_1=K_{311}})$$

$$+ \cdots + n(F_{P_1=K_{(m-1)11}})] + 3.2^{m-2},$$

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + 4.n(F_{P_1=K_{j21}})] + 6.2^{m-2},$$

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + 4.n(F_{P_1=K_{j21}})] + 3.2^{m-1}. \tag{5}$$

Now, consider

$$\sum_{j=1}^{m-1}[n(F_{P_1=K_{j21}})] = 4.\sum_{j=1}^{m-1} n(F_{P_1=K_{j21}}),$$

$$\sum_{j=1}^{m-1}[n(F_{P_1=K_{j21}})] = 4.[n(F_{P_1=K_{121}}) + n(F_{P_1=K_{221}}) + n(F_{P_1=K_{321}})$$

$$+ \cdots + n(F_{P_1=K_{(m-1)21}})]. \tag{6}$$

According to Theorem 3(ii), for every $i \in \{1, 2, ..., (m-1)\}$, we have

$$n(F_{P_1=K_{i21}}) = 2^i + (i-3)2^{i-2} = (i+1)2^{i-2}.$$

Hence, (6) can be written as

$$\sum_{j=1}^{m-1}[n(F_{P_1=K_{j21}})] = 4(1 + 3.2^0 + 4.2^1 + 5.2^2 + \cdots + m.2^{m-3})$$

$$= \frac{4}{2^3}(2.2^2 + 3.2^3 + 4.2^4 + \cdots + m.2^m)$$

$$= -1 + \frac{1}{2}\sum_{i=1}^{m} i(2^i).$$

Since $\sum_{i=1}^{m} i(2^i) = 2 + (m-1)2^{m+1}$ (see [11, p. 176]), we conclude that

$$\sum_{j=1}^{m-1} 4.n(F_{P_1=K_{j21}}) = -1 + \frac{1}{2}(2 + (m-1)2^{m+1}) = (m-1)2^m.$$

From this, we can write equation (6) as

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j22}}) + (m-1)2^m + 3.2^{m-1}$$

$$= \sum_{j=1}^{m-1} n(F_{P_1=K_{j22}}) + (2m+1)2^{m-1}. \qquad \square$$

Theorem 6. *Let G be a group satisfying Definition* 4 *with* $m \in N$, $n = s = 2$. *Then we have* $n(F_{P_1=K_{m22}}) = 2^{m-2}(m^2 + 4m + 1)$.

Proof. Consider the diagram of G (see Figure 3).

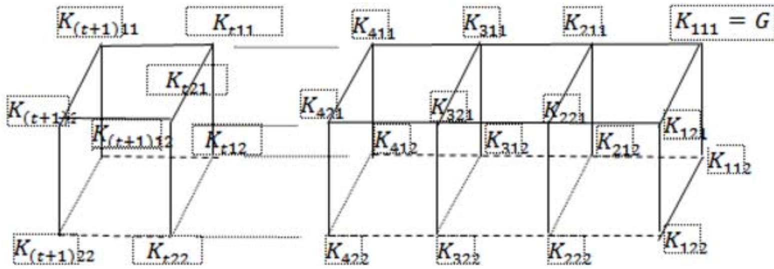

Figure 3. Lattice of cuboid group $(t+1) \times 2 \times 2$.

We prove by induction on m. For $m = 1$, it is easy to check that $n(F_{P_1 = K_{122}}) = 3$. We know that $3 = 2^{1-2}(1^2 + 4.1 + 1)$. It means that the statement is true for $m = 1$. Assume that the statement is true for $m = t$. This means that

$$n(F_{P_1 = K_{t22}}) = 2^{t-2}(t^2 + 4t + 1). \tag{7}$$

For $m = t + 1$, by considering the diagram of lattice G (see Figure 3) and applying lattice method, we get

$$n(F_{P_1 = K_{(t+1)22}}) = 2.n(F_{P_1 = K_{t22}}) + n(F_{P_1 = K_{(t+1)12}})$$
$$+ n(F_{P_1 = K_{(t+1)21}}) + n(F_{P_1 = K_{(t+1)11}}).$$

Since $n(F_{P_1 = K_{(t+1)12}}) = n(F_{P_1 = K_{(t+1)21}})$,

$$n(F_{P_1 = K_{(t+1)22}}) = 2.n(F_{P_1 = K_{t22}}) + 2.n(F_{P_1 = K_{(t+1)12}}) + n(F_{P_1 = K_{(t+1)11}}). \tag{8}$$

Substituting (7) into (8), we get

$$n(F_{P_1 = K_{(t+1)22}}) = 2.2^{t-2}(t^2 + 4t + 1) + 2.n(F_{P_1 = K_{(t+1)12}})$$
$$+ n(F_{P_1 = K_{(t+1)11}}).$$

Using Theorem 2 and Theorem 4(ii), we have

$n(F_{P_1 = K_{(t+1)11}}) = 2^{t+1} + (t-2)^{t-1}$ and $n(F_{P_1 = K_{(t+1)12}}) = 2^{t+11} + (t-2)^{t-1}$.

Hence,

$$n(F_{P_1=K_{(t+1)22}}) = 2^{t-1}(t^2+4t+1) + 2.[2^{t+1}+(t-2)2^{t-1}] + 2^{t-1},$$

$$n(F_{P_1=K_{(t+1)22}}) = 2^{t-1}(t^2+4t+1) + 2^{t+2} + (t-2)2^t + 2^{t-1},$$

$$n(F_{P_1=K_{(t+1)22}}) = 2^{(t+1)-2}(t^2+2t+1+4t+4+1),$$

$$n(F_{P_1=K_{(t+1)22}}) = 2^{(t+1)-2}[(t+1)^2+4(t+1)+1].$$

The induction is completed. □

Theorem 7. *Let G be a group satisfying Definition* 4 *with* $m \in N$, $s = 3$, $t = 2$. *Then*

$$n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j32}}) + 2.n(F_{P_1=K_{j32}})$$

$$+ 2^{m-2}(2m^2+10m+4).$$

Proof.

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} [n(F_{P_1=K_{j32}}) + n(F_{P_1=K_{j31}}) + n(F_{P_1=K_{j22}})$$

$$+ n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j12}}) + n(F_{P_1=K_{j11}})]$$

$$+ n(F_{P_1=K_{m31}}) + n(F_{P_1=K_{m21}})$$

$$+ n(F_{P_1=K_{m22}}) + n(F_{P_1=K_{m12}}). \tag{9}$$

Since $n(F_{P_1=K_{m11}}) = 2^{m-2}$, $n(F_{P_1=K_{m12}}) = n(F_{P_1=K_{m21}})$ and $n(F_{P_1=K_{i21}}) = n(F_{P_1=K_{i12}})$, equation (9) can be written as

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} [n(F_{P_1=K_{j32}}) + n(F_{P_1=K_{j31}}) + n(F_{P_1=K_{j22}})$$

$$+ n(F_{P_1=K_{j21}}) + 2.n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})]$$

$$+ n(F_{P_1=K_{m31}}) + 2.n(F_{P_1=K_{m21}}) + n(F_{P_1=K_{m32}}) + 2^{m-2}. \quad (10)$$

Since

- $n(F_{P_1=K_{m31}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j31}}) + n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})]$
  $+ n(F_{P_1=K_{m21}}) + n(F_{P_1=K_{m11}}),$
- $n(F_{P_1=K_{m21}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j21}}) + n(F_{P_1=K_{j11}})] + n(F_{P_1=K_{m11}})$ and
- $n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1}[n(F_{P_1=K_{j22}}) + n(F_{P_1=K_{j21}})] + 3.2^{m-1},$

equation (10) can be written as

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j32}})$$

$$+ \sum_{j=1}^{m-1}[2.n(F_{P_1=K_{j31}}) + 2.n(F_{P_1=K_{j22}})$$

$$+ 5.n(F_{P_1=K_{j21}}) + 4.n(F_{P_1=K_{j11}})]$$

$$+ n(F_{P_1=K_{m31}}) + 2^{m-2} + m.2^{m-1} + (2m+1)2^{m-1} + 2^{m-2},$$

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j32}})$$

$$+ \sum_{j=1}^{m-1}[2.n(F_{P_1=K_{j31}}) + 2.n(F_{P_1=K_{j22}}) + 5.n(F_{P_1=K_{j21}})]$$

$$+ 4.2^{m-2} + n(F_{P_1=K_{m21}}) + 3.2^{m-2}$$

$$+ (4m+2)2^{m-2} + 2^{m-2}. \quad (11)$$

We know that

- $\sum_{j=1}^{m-1} 4.n(F_{P_1=K_{j21}}) = (m-1)2^m$ and

- $n(F_{P_1=K_{m21}}) = 2^{m-2}$.

Therefore, equation (11) becomes

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j32}})$$

$$+ \sum_{j=1}^{m-1} [2.n(F_{P_1=K_{j31}}) + 2.n(F_{P_1=K_{j22}})]$$

$$+ 5(m-1)2^{m-1} + 2^{m-2}(m+1) + (4m+9)2^{m-2} + 2^{m-2}$$

$$= \sum_{j=1}^{m-1} n(F_{P_1=K_{j32}}) + \sum_{j=1}^{m-1} [2.n(F_{P_1=K_{j31}})$$

$$+ 2.n(F_{P_1=K_{j22}})] + 2^{m-2}(10m+6). \tag{12}$$

We can find that

- $n(F_{P_1=K_{m22}}) = \sum_{j=1}^{m-1} n(F_{P_1=K_{j22}}) + (2m+1)2^{m-1}$ and

- $\sum_{j=1}^{m-1} n(F_{P_1=K_{j22}}) = n(F_{P_1=K_{m22}}) - (2m+1)2^{m-1}$.

Therefore, equation (12) can be written as

$$n(F_{P_1=K_{m32}}) = \sum_{j=1}^{m-1} [n(F_{P_1=K_{j32}}) + 2.n(F_{P_1=K_{j31}})]$$

$$+ 2^{m-2}(2m^2 + 10m + 4). \qquad \square$$

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